Lecture 5.

Let $\mu^*: P(X) \to [0,\infty]$ be an outer measure. We want to obtain a measure by restricting to a σ-algebra.

<u>Def 1.</u> A set $E \subseteq X$ is <u>μ^*-measurable</u> if $\forall A \subseteq X$

$$(1) \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

<u>Rem.</u> By subadditivity, we always have "$\leq$" in (1). Thus, for μ^*-meas., one only needs to check

$$(2) \quad \mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

<u>Caratheodory's Thm.</u> Let μ^* be an outer measure and $\mathcal{M} = \{E \subseteq X : E \text{ is } \mu^*\text{-meas.}\}$. Then, $\mathcal{M}$ is a σ-algebra and $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure.

Pf. Clearly, $\mathcal{M}$ is closed under taking complements. To see closed under countable unions, we first check finite unions.

Let E, F be μ^*-measurable and $A \subseteq X \Rightarrow$

$$(3)\quad \mu^*(A) = \mu^*(A\cap E) + \mu^*(A\cap E^c) = \mu^*(A\cap E\cap F) + \mu^*(A\cap E\cap F^c) + \mu^*(A\cap E^c\cap F) + \mu^*(A\cap E^c\cap F^c).$$

Also, $E\cup F = (E\cap F)\cup(E\cap F^c)\cup(F\cap E^c)$.

By subadd. $\Rightarrow$

$$\mu^*(A\cap(E\cup F)) \le \mu^*(A\cap(E\cap F)) + \mu^*(A\cap(E\cap F^c)) + \mu^*(A\cap(F\cap E^c))$$

(3) $\Rightarrow$

$$(4)\quad \mu^*(A) \ge \mu^*(A\cap(E\cup F)) + \mu^*(A\cap \underbrace{E^c\cap F^c}_{(E\cup F)^c})$$

$\Rightarrow$ $E\cup F$ is μ^*-meas.

An inductive argument $\Rightarrow$ $\mathcal{M}$ is an algebra (closed under complements and finite unions).

Moreover, if $E, F \in \mathcal{M}$ are disjoint, then (4) w/ $A = E \cup F$

$$\mu^*(E \cup F) = \mu^*((E\cup F) \cap E) + \mu^*((E \cup F) \cap E^c)$$
$$= \mu^*(E) + \mu^*(F)$$

$\Rightarrow \mu = \mu^*|_{\mathcal{M}}$ is finitely additive measure.

Next, consider $E = \bigcup_{k=1}^{\infty} E_k$, where $E_k \in \mathcal{M}$.

<u>WLOG</u>. $E_k \cap E_\ell = \emptyset$ if $\ell \neq k$. (Otherwise, repeat with $E_k' = E_k \setminus \bigcup_{j=1}^{k-1} E_j$.) <u>WTS</u>. $E \in \mathcal{M}$.

Set $\underset{\text{in } \mathcal{M}}{F_k} = \bigcup_{j=1}^{k} E_k$, then $E = \bigcup_{k=1}^{\infty} F_k$ and for $A \subseteq X$:

(5) $\mu^*(A \cap F_n) = \mu^*(A \cap F_n \cap E_n) + \mu^*(A \cap F_n \cap E_n^c)$

$$= \mu^*(A \cap E_n) + \mu^*(A \cap \underbrace{\bigcup_{j=1}^{n-1} E_j}_{F_{n-1}}) = \ldots =$$

$$= \sum_{j=1}^{n} \mu^*(A \cap E_j).$$

$(5) \Rightarrow$ For any n:

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) \overset{\text{monotonicity}}{\geq}$$

$$\sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^c), \quad \text{so } n \to \infty \Rightarrow$$

$$(6)\ \mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^c) \underset{\text{subadd.}}{\geq}$$

$$\geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$\Rightarrow E \in \mathcal{M}$ as desired.

Moreover, "$\geq$" above are in fact "$=$" by remark above. Taking $A = E$ in (6) $\Rightarrow$

$$(7) \quad \mu^*(E) = \sum_{k=1}^{\infty} \mu^*(E_k) \quad \text{(countable add.)}$$

Thus, $\mathcal{M}$ is σ-algebra and $\mu = \mu^*|_{\mathcal{M}}$ is a measure. To complete proof, must establish completeness.

So, let $B \subseteq N \in \mathcal{M}$, $\mu(N) = 0$. Then, $\mu^*(B) = 0$ and for $A \subseteq X$

$$\mu^*(A) \underset{\text{subadd.}}{\leq} \underbrace{\mu^*(A \cap B)}_{=0 \text{ by monot.}} + \mu^*(A \cap B^c) \underset{\text{monot.}}{\leq} \mu^*(A) \Rightarrow B \in \mathcal{M}$$

$\Leftarrow$ μ complete. ☒